Ion-Slip Coefficients for Partially Ionized Argon and Helium

H. S. C. WANG

Bell Aerosystems Company, Buffalo, N. Y.

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Ion-slip coefficients in Demetriades and Argyropoulos' generalized Ohm's law are studied quantitatively in the first approximation for partially ionized Ar and He under various non-isothermal gas conditions. The macroscopic forces due to collisions between electrons, ion, and neutral atoms are accounted by averaging the momentum transfer cross sections over the Maxwellian velocity distributions of the colliding particles. To incorporate the appropriate interpolation and extrapolation techniques in the low energy limit, Frost and Phelps' cross sections for electron-atom collisions and Dalgarno's value for ion-atom resonant charge-transfer cross sections were used. The weighted average momentum transfer cross sections thus obtained, which are also important in other diffusion and transport processes, are tabulated at various particle kinetic temperatures for both gases. A family of curves representing values of ion-slip coefficients calculated from these data is plotted. With the generalized Ohm's law written in its inverted form, the influence of ion-slip on the "Hall conductivity" in Kruger et al's formulation is discussed.

1. Introduction

Generalized Ohm's law provides a fundamental tool for analyzing many problems involving the conduction of current in plasmas on the macroscopic basis. Early work in formulating such a relation between the current density $\boldsymbol{J}$, electric field $\boldsymbol{E}$, and magnetic induction $\boldsymbol{B}$ in a partially ionized gas is attributed to SCHLÜTER [1], COWLING [2], FINKELNBURG and MAECKER [3]. Recently DEMETRIADES and ARGYROPOULOS [4] further extended the formulation to multicomponent non-isothermal plasmas with arbitrary degree of ionization. The Ohm's law is written as

$$\boldsymbol{E}'' = \sigma^{-1} \boldsymbol{J} + \chi \boldsymbol{J} \times \boldsymbol{B} - \psi \boldsymbol{J} \times \boldsymbol{B} \times \boldsymbol{B} . \quad (1.1)$$

Here $\boldsymbol{E}''$ is the electric field $\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}$ relative to axis moving with the plasma mass velocity $\boldsymbol{U}$, plus certain terms produced by the finite gradient of electron temperature and partial pressures of plasma components. The scalar conductivity σ depends primarily on the interaction between electrons and heavy particles and has been calculated recently by SCHWEITZER and MITCHNER [5]. Due to the small electron mass, the Hall coefficient χ in the present formulation reduces to the simple expression $(n_1 e)^{-1}$. The purpose of this paper is to discuss quantitatively the ion-slip coefficient ψ from fundamental data of atomic processes for argon and helium, which are of special interest in plasma acceleration [6]. Values of ψ for both gases at different degrees of ionization were computed and plotted vs. heavy particle temperatures. For high magnetic field, the influence of ion-slip could out-weigh the improvement of accuracy in the calculation of transport coefficients to orders of approximation higher than the first [7].

2. Ion-Slip Coefficients in Terms of Basic Cross Sections of Atomic Processes

To the first order, the ion-slip coefficient for a three-component (electrons, ions, neutral particles) magneto plasma can be expressed [4] as

$$\psi = (\varrho_3/\varrho)^2/(a_{13} + a_{23}) , \quad (2.1)$$

Where the subscripts 1, 2, and 3 denote electron, ion, and neutral particle respectively, $\varrho = \sum_{i=1}^{3} \varrho_i$ is

Reprints request to H. S. C. WANG, Edo Corporation, College Point, *New York 11 356*.

1 A. SCHLÜTER, Z. Naturforsch. **5 a**, 72 [1950]; **6 a**, 73 [1951].
2 T. G. COWLING, Magnetohydrodynamics, Interscience Publishers, Inc., New York 1957.
3 W. FINKELNBURG and H. MAECKER, Elektrische Bogen und Thermisches Plasma, in: Handbuch der Physik, Band **22**, Springer-Verlag, Berlin 1956, p. 325.
4 S. T. DEMETRIADES and G. S. ARGYROPOULOS, Phys. Fluids **9**, 2136 [1966].
5 S. SCHWEITZER and M. MITCHNER, AIAA J. **4**, 1012 [1966].
6 A certain limited amount of data have been compiled by G. L. CANN et al. (in EOS Report 3160, „A Steady State Hall Current Accelerator"). However, their derivation and formulae are not compatible with the more rigorous formulation of Ref. 4, and rather rough estimates of interaction cross sections are used.
7 C. H. KRUGER, M. MITCHNER, and U. DAYBELGE, AIAA J. **6**, 1712 [1968].

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the total mass density of the plasma, $\varrho_i = n_i m_i$, n_i and m_i are respectively the mass density, the number density, the mass per particle of the i-th component of the plasma. The positive quantity e denotes the electronic charge. For Maxwellian distribution of particle velocities of each component, it can be shown * that the coefficent of interaction between i and j components is

$$a_{ij} = a_{ji} = (\tfrac{8}{3})\, \pi^{-1/2}\, n_i\, n_j\, \mu_{ij}\, \gamma_{ij}^{5/2} \int_0^\infty q_m(v)\; e^{-\gamma_{ij} v^2}\, v^5\, dv, \qquad (2.2)$$

where

$$\mu_{ij} = m_i\, m_j / (m_i + m_j),$$
$$\gamma_{ij} = \gamma_i\, \gamma_j / (\gamma_i + \gamma_j),$$
$$\gamma_i = m_i / (2\, k\, T_i),$$

k is the Boltzmann constant, T_i the kinetic temperature of the i-th component, and $q_m(v)$ denotes the momentum transfer cross section for the i-j particle collision with a relative velocity v. This formulation, when appropriately interpreted, is equivalent to Cowling's current Eq. (6.22) in his well-known tract [2].

3. Weighted Average Momentum Transfer Scattering Cross Sections

To calculate a_{ij} by (2.2), it is convenient to define a weighted average momentum transfer scattering cross section for non-isothermal plasma species i and j as

$$\overline{q_m(v)} = \gamma_{ij}^3 \int_0^\infty q_m(v)\; e^{-\gamma_{ij} v^2}\, v^5\, dv\,, \qquad (3.1)$$

which has the same dimension and units as $q_m(v)$. According to this definition, $\overline{q_m(v)}$ is related to the well-known Ω integrals of CHAPMAN and COWLING [8, 9] as

$$\overline{q_m(v)} = \{2\, \pi\, \mu_{ij} / (k\, T)\}^{1/2}\, \Omega_{ij}^{(1)}(1) \qquad (3.2)$$

when $T_i = T_j = T$ or when one particle species is considered as immobile. The Ω integrals are useful quantities and occur quite frequently in the theory of non-uniform gases [10]. To evaluate $\bar{q}_m$ in the temperature range up to 2×10^4 °K we need momentum transfer cross section data from 0 to 30 eV with those at the low energy end weighted most.

For electron-atom collision, $q_m(v)$ can be calculated by a number of methods, among which FROST and PHELPS' results are most suitable for our use [11–15]. We have computed the average momentum-transfer cross sections $\bar{q}_m$ of electrons in Ar and He by means of (3.1), using values of $q_m(v)$ calculated by a five-point Lagrange interpolation formula from the last authors' original tabulation [16]. Due to the smallness of electron mass, γ_{13} reduces to $m_1/(2\, k\, T_1)$, and $\bar{q}_m(1, 3)$ is a function of electron temperature T_1 alone. The results are summarized in Table 1. For electrons in argon, a minimum in $\bar{q}_m$ exists near 1500 °K, reminiscent of the RAMSAUER-TOWNSEND effect [17].

T_1 in 10^3 °K	$\bar{q}_{m(1,3)}$ in Ar in 10^{-16} cm^2	$\bar{q}_{m(1,3)}$ in He in 10^{-16} cm^2
0.5	0.734	5.884
1.0	0.328	6.282
1.5	0.324	6.443
2.0	0.425	6.509
3.0	0.736	6.530
4.0	1.095	6.494
5.0	1.473	6.434
6.0	1.860	6.362
7.0	2.255	6.283
8.0	2.655	6.201
9.0	3.056	6.117
10.0	3.457	6.033
12.0	4.251	5.868
14.0	5.025	5.708
16.0	5.765	5.557
18.0	6.462	5.415
20.0	7.109	5.281

Table 1. $\overline{q_{m(1,3)}}$ for electrons in Ar and He.

* This can be done, for example, by following the classical diffusion theory of LANGEVIN (Ann. Chim. Phys. **5**, 245 [1905]).

8 S. CHAPMAN and T. G. COWLING, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, Cambridge (England) 1961.

9 J. O. HIRSCHFELDER, C. F. CURTISS, and R. B. BIRD, Molecular Theory of Gases and Liquids, John Wiley & Sons, Inc., New York 1954.

10 For example, the mutual diffusion coefficient between i, j species is given in the first order but to a good approximation as

$$D_{ij} = 3\, k\, T / [16\, (n_i + n_j)\;\; \mu_{ij}\, \Omega_{ij}^{(1)}(1)]\,,$$

and the mobility of an ion at low field is related to the above via $K = e_i\, D_{ij} / (k\, T)$, where e_i denotes the ionic charge.

11 D. BARBIERE, Phys. Rev. **84**, 653 [1951].

12 T. F. O'MALLEY, Phys. Rev. **130**, 1020 [1963].

13 T. F. O'MALLEY, L. SPRUCH, and L. ROSENBERG, J. Math. Phys. **2**, 491 [1961].

14 A. DALGARNO, Diffusion and Mobilities, in: Atomic and Molecular Processes, edited by D. R. BATES (Academic Press Inc., New York 1962), Chapter 16.

15 L. S. FROST and A. V. PHELPS, Phys. Rev. **136**, 1538 [1964].

16 The author is indebted to Drs. FROST and PHELPS for kindly supplying him with a set of tabulated values of q_m for electrons in Ar and He.

17 C. RAMSAUER and R. KOLLATH, Ann. Physik **12**, 529 [1932].

For ion-atom collision, it can be shown from the theory of resonance charge transfer [18–20] that

$$q_m(v) = 2\,q(v), \quad (3.3)$$

where $q(v)$ is the total charge-exchange scattering cross section. In searching for a representative set of data for $q(v)$, it is felt that Dalgarno's values deduced from ion mobilities in their parent gases are appropriate for our purpose [21]. To interpolate and extrapolate his data, we note from Holstein's development [18] that $q_m(v)$ and hence $q(v)$ can be expressed as a quadratic in b_c, the "critical impact parameter" satisfying the relation

$$[V_1\,e^{-a b_c}/(\hbar\,v)]\cdot(2\,\pi\,b_c/a)^{1/2} = \pi/4\,,$$

where V_1 and a are constants in the charge-exchange interaction term assumed. As b_c is usually large, its dependence on the relative velocity v is very nearly of the form [const $-(\ln v)/a$]. We therefore introduce the following empirical representation

$$q(v) = A(\ln v)^2 + B\ln v + C\,. \quad (3.4)$$

With the substitution of (3.3) and (3.4) in (3.1), the integration with respect to v can be carried out to give

$$\bar{q}_m = (A/2)\,(\ln\gamma_{ij})^2 - [(\tfrac{3}{2}-\gamma)\,A + B]\,\ln\gamma_{ij} + \tfrac{1}{2}(1-3\,\gamma+\varkappa)\,A + (\tfrac{3}{2}-\gamma)\,B + 2\,C\,, \quad (3.5)$$

where $\gamma = 0.577215\ldots$ is the Euler's constant, and

$$\varkappa = \int_0^\infty (\ln x)^2\,e^{-x}\,dx = 1.978112\ldots\,.$$

The constants A, B, and C are determined from (3.4) using DALGARNO's [19] values of $q(v)$ at 0.1, 10, and 10^3 eV. Assuming equal ion and atomic masses, γ_{23} reduces to $m_3/(2\,k\,T)$, with $T = T_2 + T_3$. Values of $\bar{q}_{m(2,3)}$ computed for Ar and He are summarized in Table 2.

4. Numerical Computation of Ion-Slip Coefficient ψ

Utilizing the average momentum transfer cross section $\bar{q}_m$ computed in the previous section, the calculation of ψ from (2.1) and (2.2) is straightforward. A typical family of curves showing the variation of ψ with T ($a_{13} \ll a_{23}$) is displayed in

$T = T_2 + T_3$ in 10^3°K	$\bar{q}_{m(2,3)}$ for Ar^+ in Ar in 10^{-16} cm²	$\bar{q}_{m(2,3)}$ for He^+ in He in 10^{-16} cm²
0.5	153.5	76.6
1.0	147.3	71.7
2.0	141.1	67.0
3.0	137.6	64.4
4.0	135.1	62.5
5.0	133.2	61.1
6.0	131.6	60.0
7.0	130.3	59.0
8.0	129.2	58.2
9.0	128.2	57.5
10.0	127.3	56.8
12.0	125.8	55.7
14.0	124.5	54.8
16.0	123.4	54.0
18.0	122.4	53.3
20.0	121.5	52.7

Table 2. $\overline{q_{m(2,3)}}$ for ions in parent gas.

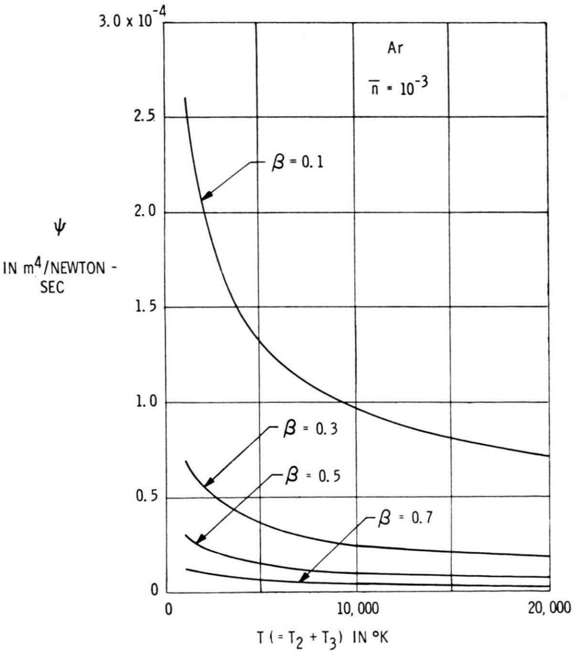

Fig. 1. Ion-Slip coefficent ψ for partially ionized Argon.

[18] T. HOLSTEIN, J. Phys. Chem. **56**, 832 [1952].
[19] A. DALGARNO, Phil. Trans. Roy. Soc. London A **250**, 426 [1958].
[20] E. A. MASON, J. T. VANDERSLICE, and J. M. YOS, Phys. Fluids **2**, 688 [1959].
[21] For He^+ in He, DALGARNO [19] assumed a resonance interaction of $4.92\,re^{-1.40\,r}$ Rydbergs, and was able to match both the high energy charge-transfer cross sections of GILBODY and HASTED [22] and the mobility values of HORNBECK [23], BIONDI and CHANIN [24] at 300 °K. A similar procedure is also applied for Ar^+ in Ar to reproduce Hornbeck, Biondi and Chanin's ion mobility data at 300 °K.
[22] H. B. GILBODY and J. B. HASTED, Proc. Roy. Soc. London A **238**, 334 [1957].
[23] J. A. HORNBECK, Phys. Rev. **83**, 374; **84**, 621 [1951].
[24] M. A. BIONDI and L. M. CHANIN, Phys. Rev. **94**, 910 [1954]; **106**, 473 [1957].

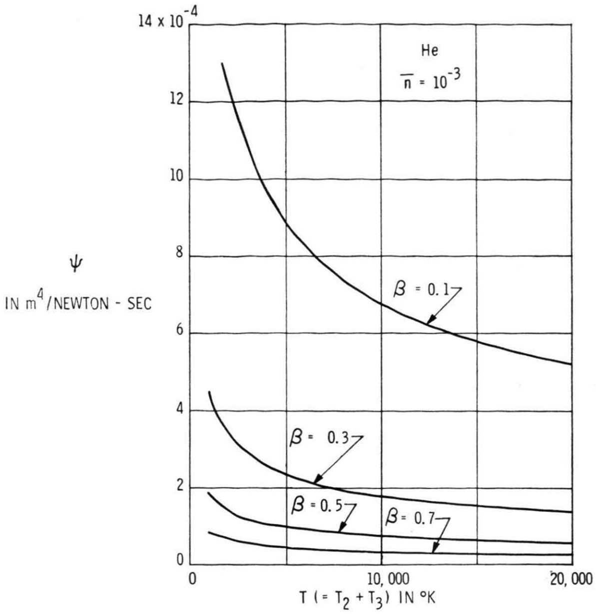

Fig. 2. Ion-Slip coefficent ψ for partially ionized Helium.

Figs. 1 and 2 for Ar and He respectively, where the parameter $\beta = n_2/(n_2+n_3)$ denotes the degree of ionization, and $\bar{n} = (n_2+n_3)/n_0$ is the total heavy particle number density expressed as a fraction of the Loschmidt's number $n_0 = 2.687 \times 10^{19}\ \text{cm}^{-3}$.

It is interesting to note that (1.1) can be inverted to give $\boldsymbol{J}$ in terms of $\mathbf{E}''$ (l. c. [4]) :

$$\boldsymbol{J} = \sigma \left\{ \mathbf{E}'' - \frac{\sigma \chi}{(1+\sigma \psi B^2)^2 + (\sigma \chi B)^2} \mathbf{E}'' \times \boldsymbol{B} + \frac{\sigma^2 \chi^2 + \sigma \psi (1+\sigma \psi B^2)}{(1+\sigma \psi B^2)^2 + (\sigma \chi B)^2} \mathbf{E}'' \times \boldsymbol{B} \times \boldsymbol{B} \right\}. \quad (4.1)$$

If ion-slip is neglected or $\psi = 0$, the coefficient of the $\mathbf{E}'' \times \boldsymbol{B}$ term reduces to the "Hall Conductivity" $\sigma_{\mathrm{H}}^{(1)}$ of KRUGER, MITCHNER, and DAYBELGE [7], and the coefficient of the $\mathbf{E}'' \times \boldsymbol{B} \times \boldsymbol{B}$ term constitutes part of their $\sigma_{\perp}^{(1)}$. When $B \geqq [a_{23}/(n_1 e)]\,(\varrho/\varrho_3)^2$, say of the order of 1 weber/m^2 in the range of parameters displayed in Figs. 1 and 2, the influence of ion-slip on transport properties could become more important than the improvement of accuracy by carrying calculations to second and higher orders of approximation.

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